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THE WIENER-HOPF TECHNIQUE & DISCRETELY MONITORED PATH DEPENDENT OPTION PRICING

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Abstract. Fusai et al. (2006) employed the Wiener-Hopf technique to obtain an exact analytic expression for discretely monitored barrier option prices as the solution to the Black-Scholes partial differential equation (PDE). The present work reformulates this in the language of random walks and extends it to price a variety of other discretely monitored path dependent options. Analytic arguments familiar in the applied mathematics literature are used to obtain fluctuation identities. This includes casting the famous identities of Baxter and Spitzer in a form convenient to price barrier, first-touch and hindsight options. Analyzing random walks killed by two absorbing barriers with a modified Wiener-Hopf technique yields a novel formula for double-barrier option prices. Continuum limits and continuity correction approximations are considered. Numerically efficient results are obtained by implementing Padé approximation. A Gaussian Black-Scholes framework is used as a simple model to exemplify the techniques, but the analysis applies to Lévy processes generally.

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1. Introduction

The application of complex analytic techniques within probability theory has a long and distinguished history. Yet, convenient probabilistic identities in their modern formulation often hide deep analytic structure from which simple formulae and computational methods can be developed. This work describes the application of transform techniques and Wiener-Hopf type arguments to price a variety of discrete path dependent financial contracts, including barrier, first-touch, hindsight and double-barrier options. Auxiliary derivation of known identities and simple asymptotic analysis show explicitly the relationship to other semi-analytical and approximation techniques. For simplicity, the Black-Scholes model is used to exemplify results throughout, though analysis applies equally well to Lévy processes.

The use of transformation techniques in financial mathematics is quite modern, see Carr and Madan (1998). Their introduction coincides with the increased popularity of models for which the characteristic functions (Fourier transformed probability densities) generally have simpler form than densities themselves and efficient computation may be achieved using the Fast Fourier Transform. Such models include Lévy processes, stochastic volatility models and affine models in the interest rate literature. A recent investigation into the analytic properties of characteristic functions, transformed payoffs and the calculation of 'vanilla' option prices in Fourier space using Parseval's identity is discussed by Lewis (2001).

The Wiener-Hopf technique was devised by Norbert Wiener and Eberhard Hopf in 1931 to exactly solve certain integral equations where the domain of integration is restricted to the half-line. By using the properties of analytic functions they extended the applicability of Fourier transformation techniques. The Wiener-Hopf method remains an enduring tool in analysis. Briefly, Fourier transformation results in a Riemann-Hilbert equation defined in a strip in the complex transform parameter plane. The key step in the procedure is then to factorize the Fourier transform of the integration kernel (transition density) into a product of two functions that are analytic and algebraic above/below the strip in which the kernel is analytic. A solution is then provided by a sophisticated and elegant function theoretic argument involving analytic continuation and Liouville's theorem. For full details the reader is referred to Noble (1988). There are few direct applications of the Wiener-Hopf technique in the finance literature, but expressions for the prices of several path-dependent options under Lévy processes with continuous monitoring are obtained in Boyarchenko and Levendorskii (2002) using Wiener-Hopf factorizations and pseudodifferential analysis.

The contractual details for many path dependent options specify a contingency upon the underlying assets at discrete times, often daily closing. The often substantial price discrepancies between options with discrete monitoring, and those calculated assuming option payoffs are dependent on functionals of continuous time paths, are well recognized. Many approximations and numerical approaches have been proposed to overcome the misprising, for a review containing a list of relevant articles, see Fusai and Recchioni (2007). Yet, fast reliable calculation of accurate prices for these contracts continues to present a significant challenge.

The fluctuation of discrete time random walks is a natural framework to study discretely monitored options. Using Bohnenblust's combinatorial algorithm Spitzer (1956) obtained a spectacular result connecting the distribution of the maximum of partial sums of independent identically distributed random variables (a random walk) to the distribution of their positive partial sums. The result was known previously to Pollaczek (1952) in terms of the solution to a singular integral equation arising in his work in queueing theory. Subsequently, Baxter (1961) derived Spitzer's and other results (concerning a random walk's exit from the half line) in an algebraic manner closely related to the Wiener-Hopf technique. Further informative discussion on the relationship between fluctuation identities and Wiener-Hopf factorization may be found in, for example, Borovkov (1973) or Feller (1971). The exit of a random walk from a finite interval is analysed by a Wiener-Hopf type technique in Kemperman (1963).

Fusai et al. (2006) solved Black-Scholes PDE to obtain an exact analytic expression for the prices of discretely monitored down-and-out European call options. This involved reducing the PDE to a series of nested integrals expressed as a recurrence relation in the discounted price between successive monitoring dates. Taking the z-transform (generating function) gave a Fredholm integral equation of the second kind with convolution structure which was solved by employing Fourier transforms and the Wiener-Hopf technique. The Fourier inverse was evaluated by residue calculus. Utilizing the algorithm introduced by Abate and Whitt (1992) to approximate the inverse z-transform, good numerical accuracy and computational efficiency were achieved.

The present work reformulates and extends the results of Fusai et al. (2006) by first considering various *probabilistic identities* for random walks. These concern their killing by barriers, when they cross barriers and their extrema. Included is a derivation of Spitzer's and Baxter's identities in an analytically tractable form. The modified Wiener-Hopf technique (commonly referred to as Jones' method, see Noble (1988)) provides a novel analytic expression for the (transformed) distribution of a random walk killed by two absorbing barriers. Utilizing these results, *pricing formulae* for discretely monitored *barrier*, *first touch*, *hindsight* and *double barrier* options are then presented in Section 3. Working in Fourier space using Parseval's identity the method also extends results described in Lewis (2001) to price 'exotic' contracts. Simple *asymptotic analysis* in Section 4 gives analogous formulae in continuous time and provides direct and explicit derivation of so called 'continuity correction approximations'. *Numerical results* are discussed in Section 5, where a very accurate and efficient method employing Padé approximation is shown to be superior to alternative methods. The closing discussion concerns ongoing work to explore and extend this methodology.

2. Probabilistic Identities

2.1. **Barrier Absorption.** Consider a random walk $R_n = R_{n-1} + X_n,\ R_0 = 0$, where X_n are given independent identically distributed random variables with law $k(x)$. This traces out a path where the transition probability that $R_{n+1} = x$ while $R_n = y$ has density $k(x-y)$, i.e.,

$$k(x-y)dx = \mathbb{P}\left(\{R_{n+1} \in [x, x+dx)\} | \{R_n = y\}\right), \tag{2.1}$$

where $\mathbb{P}$ denotes a probability measure on path-space.

2.1.1. *Lower Barrier.* Denote by $\wp_n^{(\mathrm{d})}(x)$ the (defective) probability density that $R_n = x$ is realized by a path that does not cross a barrier $d < 0$,

$$\wp_n^{(\mathrm{d})}(x)dx = \mathbb{P}\left(\{R_n \in [x, x+dx)\} \cap \{\underline{R}_{n-1} \geqslant d\}\right), \tag{2.2}$$

where $\underline{R}_n = \min_{0 \leqslant j \leqslant n} \{R_j\}$, then since

$$\underline{R}_n = \min\{\underline{R}_{n-1}, R_n\} = \min\{\underline{R}_{n-1}, R_{n-1} + X_n\}, \tag{2.3}$$

$$\wp_n^{(\mathrm{d})}(y) = \int_d^\infty \wp_{n-1}^{(\mathrm{d})}(x)k(y-x)dx, \;\; \wp_0^{(\mathrm{d})}(x) = \delta(x). \tag{2.4}$$

Here $\delta(x)$ is the Dirac delta function. Taking the *z-transform* (generating function) of $\wp_n^{(\mathrm{d})}(y)$,

$$f(y,q) = \mathcal{Z}\wp_n^{(\mathrm{d})}(y) = \sum_{n=0}^{\infty} q^n \wp_n^{(\mathrm{d})}(y), \tag{2.5}$$

gives

$$f(y,q) - \delta(y) = q\int_d^\infty f(x,q)k(y-x)dx. \tag{2.6}$$

The z-transform exists within a disc $\mathcal{D}$ in the q-plane which, by Abel's theorem for power series with non-negative coefficients, has radius $\geqslant 1$.

A change of variables ($\xi = x-d$, $\nu = y-d$) and defining $g(\xi) = f(\xi+d,q)$ gives

$$g(\nu) - \delta(\nu+d) = q\int_0^\infty g(\xi)k(\nu-\xi)d\xi. \tag{2.7}$$

With the *Fourier transform* (characteristic function) defined

$$K(z) = \mathcal{F}k(x) = \int_{-\infty}^{\infty} e^{iz\xi}k(\xi)d\xi \tag{2.8}$$

and half-range transforms

$$G_-(z) = \int_{-\infty}^0 e^{iz\xi}g(\xi)d\xi, \;\; G_+(z) = \int_0^\infty e^{iz\xi}g(\xi)d\xi, \tag{2.9}$$

application of the convolution theorem gives

$$G_+(z)L(z) = e^{-izd} - G_-(z), \tag{2.10}$$

where

$$L(z) = 1 - qK(z). \tag{2.11}$$

On the assumption $e^{-\nu_{\mp}\xi}g(\xi) \to 0$ ($\nu_- < 0 < \nu_+$) as $\xi \to \pm\infty$, equation (2.10) is valid in a strip $\mathcal{S}$ around the origin in which $K(z)$ is analytic (for general analyticity properties of characteristic functions see Lukacs and Szasz (1953)) and $1 - qK(z) \neq 0$. The subscript notation $\pm$ refers here and henceforth to functions which are analytic and converge to constant value as $|z| \to \infty$ in and above/below $\mathcal{S}$. Introducing the *product factorization* $L(z) = L_-(z)L_+(z)$ of $L(z) = 1-qK(z)$ (setting $L_\pm(z) \to 1$ as $|z| \to \infty$ in $\mathcal{S}$) and the *sum factorization* $P(z) = P_-(z) + P_+(z)$ of

$$P(z) = e^{-idz}/L_-(z) \tag{2.12}$$

where $P_\pm(z) \to 0$ as $|z| \to \infty$ in $\mathcal{S}$, equation (2.10) becomes

$$G_+(z)L_+(z) - P_+(z) = P_-(z) - G_-(z)/L_-(z). \tag{2.13}$$

This equates (within $\mathcal{S}$) a function analytic in and above $\mathcal{S}$ (i.e. a '+'-function) to a function analytic in and below $\mathcal{S}$ ('$-$'-function), so by *analytic continuation* defines an entire function.

On the assumption $g(\xi) = f(\xi+d)$ is bounded at $\xi = 0$, $G_\pm(z) \sim 1/z$ as $|z| \to \infty$, so by *Liouville's theorem* (2.13) equates to zero, giving

$$G_+(z) = \frac{P_+(z)}{L_+(z)} \tag{2.14a}$$

$$= \frac{e^{-idz}}{L(z)} - \frac{P_-(z)L_-(z)}{L(z)}. \tag{2.14b}$$

Explicit integral representations for the Wiener-Hopf factors are

$$P_\pm(z) = \pm\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{-id\zeta}}{L_-(\zeta)(\zeta - z)} d\zeta \quad (\Im\zeta \lesseqgtr \Im z)\,, \tag{2.15}$$

and

$$L_\pm(z) = \exp\left\{\pm\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\ln L(\zeta)}{\zeta - z} d\zeta\right\} \quad (\Im\zeta \lesseqgtr \Im z)\,. \tag{2.16}$$

The probability densities $\wp_n^{(\mathrm{d})}(x)$ are recovered as

$$\wp_n^{(\mathrm{d})}(x) = \mathcal{Z}_n^{-1} g(x - d) = \begin{cases} \mathcal{Z}_n^{-1}\mathcal{F}^{-1} e^{idz} G_+(z) & \text{for } x > d, \\ \mathcal{Z}_n^{-1}\mathcal{F}^{-1} e^{idz} G_-(z) & \text{for } x < d. \end{cases} \tag{2.17}$$

On substitution, for $x > d$,

$$\wp_n^{(\mathrm{d})}(x) = \mathcal{Z}_n^{-1}\mathcal{F}^{-1}\left(\frac{1}{L(z)} - e^{idz}\frac{P_-(z)L_-(z)}{L(z)}\right). \tag{2.18}$$

The first term in (2.18) may be identified as $\wp_n(x)$, the probability density without barriers, defined in (2.4) with $d \to \infty$. Hence,

$$\boxed{\wp_n^{(\mathrm{d})}(x) = \wp_n(x) - \frac{\mathcal{Z}_n^{-1}}{2\pi}\left(\int_{-\infty}^{\infty} \frac{P_-(z)L_-(z)}{L(z)} e^{-i(x-d)z} dz\right).} \tag{2.19}$$

The inverse z-transform $\mathcal{Z}_n^{-1}$ is the Taylor coefficient of q^n, which is given by

$$\mathcal{Z}_n^{-1} f(q) = \oint_\Lambda f(q) q^{-(n+1)} dq, \tag{2.20}$$

where $\Lambda \in \mathcal{D}$ is a simply connected contour enclosing $q = 0$.

The probability that the barrier has not been crossed is

$$\mathbb{P}\left(\{\underline{R}_n \geqslant d\}\right) = \int_d^\infty \wp_n^{(\mathrm{d})}(x) dx. \tag{2.21}$$

2.1.2. *Upper Barrier.* With analogous reasoning to that of the previous subsection, denote by $\wp_n^{(\mathrm{u})}(x)$ the probability density that $R_n = x$ is realized by a path that does not cross a barrier $u > 0$, that is

$$\wp_n^{(\mathrm{u})}(x) dx = \mathbb{P}\left(\{R_n \in [x, x + dx)\} \cap \{\bar{R}_{n-1} \leqslant u\}\right), \tag{2.22}$$

where $\bar{R}_n = \max_{0 \leqslant j \leqslant n} \{R_j\}$, then

$$\boxed{\wp_n^{(\mathrm{u})}(x) = \wp_n(x) - \frac{\mathcal{Z}_n^{-1}}{2\pi} \left(\int_{-\infty}^{\infty} \frac{Q_+(z) L_+(z)}{L(z)} e^{-i(x-u)z} dz \right).} \tag{2.23}$$

Here $L_+(z)$ is again given by (2.16) and

$$Q_+(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{-iu\zeta}}{L_+(\zeta)(\zeta - z)} d\zeta \quad (\Im\zeta < \Im z). \tag{2.24}$$

2.2. **Barrier First-Crossing.** The first hitting time τ_d of the barrier $d < 0$ is the first occasion $R_n < d$:

$$\tau_d = \min\{j \geqslant 1 \,|\, R_j < d\}, \tag{2.25}$$

with associated overshoot $d - R_{\tau_d}$.

The probability that n is the first hitting time is

$$\mathbb{P}(\{\tau_d = n\}) = \int_{-\infty}^{d} \wp_n(x) dx = G_-(0). \tag{2.26}$$

Clearly

$$\mathbb{P}(\{\tau_d = n\}) = \mathbb{P}(\{\underline{R}_n \geqslant d\}) - \mathbb{P}(\{\underline{R}_{n-1} \geqslant d\}), \tag{2.27}$$

that is, for $x < d$,

$$\wp_n^{(\mathrm{d})}(x) dx = \mathbb{P}(\{R_n \in [x, x+dx)\} \cap \{\tau_d = n\}). \tag{2.28}$$

2.2.1. *Analytic Continuation For $R_n < d$.* Referring back to (2.17) and (2.13), analytic continuation for $R_n < d$ provides

$$G_-(z) = L_-(z) P_-(z) \tag{2.29}$$

and, for $x < d$,

$$\boxed{\wp_n^{(\mathrm{d})}(x) = \frac{\mathcal{Z}_n^{-1}}{2\pi} \left(\int_{-\infty}^{\infty} P_-(z) L_-(z) e^{-i(x-d)z} dz \right).} \tag{2.30}$$

2.2.2. *Baxter's Identity.* The first (strict) ascending ladder epoch τ^+ is the entrance time into $(0, \infty)$, that is

$$\tau^+ = \min\{j \geqslant 1 \,|\, R_j > 0\}. \tag{2.31}$$

The associated first ascending ladder height is S_{τ^+}. Correspondingly, the first (weak) descending ladder epoch is the entrance time into $(-\infty, 0]$, given by

$$\tau^- = \min\{j \geqslant 1 \,|\, R_j \leqslant 0\}, \tag{2.32}$$

and first descending ladder height is S_{τ_-}.

Notice that with $d = 0$, $G_-(z)$ is the Fourier z-transform of the joint probability density of the first crossing time τ^- into $(0, -\infty)$ and the first descending ladder height R_{τ^-},

$$\mathcal{ZF}\left[\mathbb{P}\left(\{R_{\tau^-} \in [x, x+dx)\} \cap \{\tau^- = n\}\right) / \, dx\right] = G_-(z). \tag{2.33}$$

The composition of $G_+(z)$ includes $\delta(x)$, but $G_-(z)$ remains bounded as $x \to 0$ and so both sides of (2.13) (with $P_-(z) = 1/L_-(z)$ and $P_+(z) = 0$) are equal to 1, that is,

$$1 - G_-(z) = L_-(z). \tag{2.34}$$

This relation represents a famous Wiener-Hopf factorization in probability theory implicit in Baxter's identity,

$$\boxed{1 - \mathbb{E}\left[q^{\tau^+} e^{izR_{\tau^+}}\right] = \exp\left\{-\sum_{n=1}^{\infty} \frac{q^n}{n} \mathbb{E}\left(e^{izR_n} \middle| R_n > 0\right)\right\}.} \tag{2.35}$$

Given $\mathbb{E}\left(e^{izR_n} \middle| R_n > 0\right)$ is the half-range Fourier transform of the probability density function for the random walk R_n, (2.35) must have the properties of a '+'-function. To confirm (2.35) as a Wiener-Hopf factor,

$$\sum_{n=1}^{\infty} \frac{q^n}{n} \left[\mathbb{E}\left[e^{izR_n} \middle| R_n > 0\right] + \mathbb{E}\left[e^{izR_n} \middle| R_n \leqslant 0\right]\right] = \sum_{n=1}^{\infty} \frac{q^n}{n} \mathbb{E}\left[e^{izR_n}\right] \tag{2.36}$$

and recalling X_j have independent identical distributions,

$$\sum_{n=1}^{\infty} \frac{q^n}{n} \mathbb{E}\left[e^{izR_n}\right] = \sum_{n=1}^{\infty} \frac{q^n}{n} \prod_{j=1}^{n} \mathbb{E}\left[e^{izX_n}\right] \tag{2.37a}$$

$$= \sum_{n=1}^{\infty} \frac{q^n}{n} \left(\mathbb{E}\left[e^{izX_j}\right]\right)^n \tag{2.37b}$$

$$= -\ln\left(1 - qK(z)\right), \tag{2.37c}$$

where the final step used the identity $\sum_{n=1}^{\infty} q^n/n = -\ln(1-q)$.

2.3. **Partial Maximum.** Observing $\bar{R}_n$ has the same distribution as Q_n, where

$$Q_0 = 0 \ : \ Q_n = (Q_{n-1} + X_n)^+, \tag{2.38}$$

the probability densities $\wp_n^{(\mathrm{m})}(x)$ for $\bar{R}_n$ must satisfy

$$\wp_{n+1}^{(\mathrm{m})}(y) = \int_0^{\infty} \wp_n^{(\mathrm{m})}(x) k(y-x) dx. \tag{2.39}$$

Repeating the previous Wiener-Hopf analysis yields an expression for $H_+(z) = \mathcal{ZF}\wp_n^{(\mathrm{m})}(x)$,

$$H_+(z) = \frac{1}{L_-(0) L_+(z)}, \tag{2.40}$$

so that

$$\boxed{\wp_n^{(\mathrm{m})}(x) = \frac{\mathcal{Z}_n^{-1}}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-ixz}}{L_-(0)L_+(z)} dz.} \tag{2.41}$$

Alternatively, since

$$\wp_n^{(\mathrm{m})}(u) = \frac{\partial}{\partial u} \mathbb{P}\left(\{\bar{R}_n < u\}\right) = \frac{\partial}{\partial u} \int_{-\infty}^{\infty} \wp_n^{(\mathrm{u})}(x) dx, \tag{2.42}$$

using the identity $\int_{-\infty}^{\infty} \left[\mathcal{F}^{-1}F\right](x)dx = F(0)$, (2.40) may be obtained directly from (2.23),

$$\begin{aligned}
\wp_n^{(\mathrm{m})}(u) &= \frac{\partial}{\partial u} \int_{-\infty}^{\infty} \left(\wp_n(x) - \mathcal{Z}_n^{-1} \mathcal{F}^{-1} \left[\frac{Q_+(z)}{L_-(z)} e^{iuz} \right] \right) dx && (2.43a)\\
&= -\mathcal{Z}_n^{-1} \frac{\partial}{\partial u} \left[\frac{Q_+(0)}{L_-(0)} \right] && (2.43b)\\
&= \frac{-\mathcal{Z}_n^{-1}}{L_-(0)} \frac{1}{2\pi i} \frac{\partial}{\partial u} \int_{-\infty}^{\infty} \frac{e^{-iuz}}{L_+(z)z} dz && (2.43c)\\
&= \mathcal{Z}_n^{-1} \mathcal{F}^{-1} \left[\frac{1}{L_-(0)L_+(z)} \right]. && (2.43d)
\end{aligned}$$

Equation (2.40) represents the Wiener-Hopf factorization implicit in Spitzer's identity

$$\boxed{\sum_{n=0}^{\infty} q^n \mathbb{E}\left[e^{iz\bar{R}_n}\right] = \exp\left\{ \sum_{n=1}^{\infty} \frac{q^n}{n} \mathbb{E}\left[e^{izR_n^+}\right] \right\}}. \tag{2.44}$$

Normalizing the z-transform, the expectation of e^{izR_n} geometrically distributed with parameter q is

$$\begin{aligned}
\mathbb{E}_q\left[e^{izR_n}\right] &= (1-q) \sum_{n=0}^{\infty} q^n \mathbb{E}\left[e^{izR_n}\right] && (2.45a)\\
&= (1-q) \sum_{n=0}^{\infty} q^n \prod_{j=1}^{n} \mathbb{E}\left[e^{izX_j}\right] && (2.45b)\\
&= (1-q) \sum_{n=0}^{\infty} q^n \left(\mathbb{E}\left[e^{izX_j}\right] \right)^n && (2.45c)\\
&= \frac{1-q}{1-q\mathbb{E}\left[e^{izX_j}\right]} = \frac{L(0)}{L(z)}, && (2.45d)
\end{aligned}$$

which makes clear the factorization

$$\mathbb{E}_q\left[e^{izR_n}\right] = \mathbb{E}_q\left[e^{iz\underline{R}_n}\right] \mathbb{E}_q\left[e^{iz\bar{R}_n}\right]. \tag{2.46}$$

2.4. **Double-Barrier Absorption.** Now denote by $\wp_n^{(\mathrm{ud})}(x)$ the probability density that $R_n = x$ is realized by a path that crosses neither a barrier $d < 0$ nor $u > 0$. Then

$$\wp_{n+1}^{(\mathrm{ud})}(y) = \int_d^u \wp_n^{(\mathrm{ud})}(x) k(y-x) dx, \;\; \wp_0^{(\mathrm{ud})}(x) = \delta(x). \tag{2.47}$$

Once more, z-transform $f(x,q) = \mathcal{Z}\wp_n^{(\mathrm{ud})}(x)$,

$$f(y,q) - \delta(y) = q\int_d^u f(x,q)k(y-x)dx. \tag{2.48}$$

Let $\zeta = y-d$, $\eta = x-d$ and $j(\zeta) = f(\zeta+d,q)$, then

$$j(\zeta) - \delta(\zeta+d) = q\int_0^{u-d} j(\eta)k(\zeta-\eta)d\eta. \tag{2.49}$$

Invoking the convolution theorem while Fourier transforming this equation yields

$$J_-(z) + L(z)J_0(z) + e^{i(u-d)z}J_+(z) = e^{-idz}, \tag{2.50}$$

where

$$J_-(z) = \int_{-\infty}^0 j(\zeta)e^{iz\zeta}d\zeta, \tag{2.51a}$$

$$J_0(z) = \int_0^{u-d} j(\zeta)e^{iz\zeta}d\zeta = e^{(u-d)iz}\int_{d-u}^0 j(\zeta+u-d)e^{iz\zeta}d\zeta, \tag{2.51b}$$

$$J_+(z) = \int_0^\infty j(\zeta+u-d)e^{iz\zeta}d\zeta. \tag{2.51c}$$

Introduce the product factorization $L(z) = L_+(z)L_-(z)$ and divide by $L_-(z)$ and $e^{i(u-d)}L_+(z)$ to give

$$\frac{J_-(z)}{L_-(z)} + L_+(z)J_0(z) + \frac{e^{i(u-d)z}J_+(z)}{L_-(z)} = \frac{e^{-idz}}{L_-(z)} \tag{2.52}$$

and

$$\frac{e^{-i(u-d)z}J_-(z)}{L_+(z)} + L_-(z)e^{-i(u-d)z}J_0(z) + \frac{J_+(z)}{L_+(z)} = \frac{e^{-iuz}}{L_+(z)} \tag{2.53}$$

respectively. These are simplified by utilizing the sum factorizations

$$\tilde{P}(z) = \tilde{P}_+(z) + \tilde{P}_-(z), \qquad \tilde{P}(z) = \frac{e^{-idz}}{L_-(z)} - \frac{e^{i(u-d)z}J_+(z)}{L_-(z)}, \tag{2.54a}$$

$$\tilde{Q}(z) = \tilde{Q}_+(z) + \tilde{Q}_-(z), \qquad \tilde{Q}(z) = \frac{e^{-iuz}}{L_+(z)} - \frac{e^{-i(u-d)z}J_-(z)}{L_+(z)} \tag{2.54b}$$

and rearranging to give

$$\frac{J_-(z)}{L_-(z)} - \tilde{P}_-(z) = \tilde{P}_+(z) - L_+(z)J_0(z) \tag{2.55}$$

and

$$\frac{J_+(z)}{L_+(z)} - \tilde{Q}_+(z) = \tilde{Q}_-(z) - L_-(z)e^{-i(u-d)z}J_0(z). \tag{2.56}$$

As before these equate functions analytic and vanishing as $|z| \to \infty$ in the upper half-plane to functions with these properties in the lower half-plane, so by analytic continuation define entire functions. Note that $J_0(z)$ is itself an entire function. By application of Liouville's theorem entire functions (2.55) and (2.56) are zero.

Explicit integral representations for $\tilde{P}_-(z)$ and $\tilde{Q}_+(z)$ yield

$$\frac{J_-(z)}{L_-(z)} - \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{i(u-d)\zeta} J_+(\zeta)}{(\zeta - z)L_-(\zeta)} d\zeta = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{-id\zeta}}{(\zeta - z)L_-(\zeta)} d\zeta \quad (\Im\zeta > \Im z) \tag{2.57}$$

and

$$\frac{J_+(z)}{L_+(z)} + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{-i(u-d)\zeta} J_-(\zeta)}{(\zeta - z)L_+(\zeta)} d\zeta = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{-iu\zeta}}{(\zeta - z)L_+(\zeta)} d\zeta \quad (\Im\zeta < \Im z). \tag{2.58}$$

The interaction between the boundaries is resolved by finding values of $J_+(z)$ and $J_-(z)$ from these coupled integral equations. In the Gaussian case, where $1/L(z)$ is meromorphic with infinite sets of simple poles above and below the analytic strip, an approximate solution is attainable. This is achieved by solving the system of linear equations given by truncating the infinite sums of residues resulting from deforming (see Appendix A3) the contours in (2.57) upwards and those in (2.58) downwards. However, when either u or d become small, convergence of the sums becomes slow and accurate approximation demands the inversion of large matrices. As detailed in Section 5.1.3, this drawback may be avoided using rational approximations to $L_\pm(z)$.

Re-arranging (2.50) to give $\mathcal{ZF}\wp_n^{(\mathrm{ud})}(x) = e^{idz} J_0(z)$ for $d < x < u$,

$$\mathcal{ZF}\wp_n^{(\mathrm{ud})}(x) = \frac{1}{L(z)} - \frac{e^{idz} J_-(z)}{L(z)} - \frac{e^{iuz} J_+(z)}{L(z)}. \tag{2.59}$$

This may be interpreted as a generalized form of Wald's identify, see Miller (1961). Taking the inverse transforms gives

$$\boxed{\wp_n^{(\mathrm{ud})}(x) = \wp_n(x) - \frac{\mathcal{Z}_n^{-1}}{2\pi}\int_{-\infty}^{\infty} \left(\frac{e^{idz} J_-(z)}{L(z)} + \frac{e^{iuz} J_+(z)}{L(z)} \right) e^{-ixz} dz,} \tag{2.60}$$

where $\wp_n(x)$ is the probability density without barriers. Notice that as $u \to \infty$, $J_0(z) \to G_+(z)$, $\tilde{P}(z) \to P(z)$, $J_-(z) \to P_-(z)L_-(z)$ and, for $\Im z > 0$, $e^{iuz} \to 0$. Thus the single barrier results are recovered as $d \to -\infty$ or $u \to \infty$.

2.5. **Double-Barrier First-Crossing.** The Fourier z-transform of the probability density for the random walk's first crossing over either barrier $x < d$ or $x > u$ is given by (the analytic continuation) $e^{idz} J_-(z) + e^{iuz} J_+(z)$, so that

$$\boxed{\wp_n^{(\mathrm{ud})}(x) = \wp_n(x) - \frac{\mathcal{Z}_n^{-1}}{2\pi}\int_{-\infty}^{\infty} \left(J_-(z) e^{-i(x-d)z} + J_+(z) e^{-i(x-u)z} \right) dz.} \tag{2.61}$$

for $x < d$ or $x > u$.

3. Pricing Formulae

Derivatives are priced as the discounted expected payoff under a specified risk-neutral probability measure $\mathbb{Q}$ (equivalent to the statistical probability measure $\mathbb{P}$). Suppose claims are contingent upon an underlying asset $S_t = S_0 e^{R_t}$ $(t \geqslant 0)$. See Appendix A1 for the formulae for vanilla option prices in Fourier space. For path dependent options, monitored discretely at increments Δt the log price process is effectively a random walk $R_n \equiv R_{n\Delta t}$.

For simplicity, numerical examples (in Section 5) assume exponential Brownian motion, making $\mathbb{Q}$ unique. The random walk R_n then has normally distributed increments $\mathscr{N}\left[\left(r - \frac{\sigma^2}{2}\right)\Delta t, \sigma^2 \Delta t\right]$ for risk free rate r, volatility σ and intermonitoring time Δt. The transition probability density is

$$k(x) = \frac{1}{\sqrt{2\pi\sigma^2\Delta t}} \exp\left[-\frac{\left(x - \left(r - \frac{\sigma^2}{2}\right)\Delta t\right)^2}{2\sigma^2\Delta t}\right], \tag{3.1}$$

so that the Fourier transform (characteristic function) of the price increments in the underlying asset is

$$K(z) = \exp\left\{iz\left(r - \sigma^2/2\right)\Delta t - z^2\sigma^2\Delta t/2\right\}, \tag{3.2}$$

valid for all z.

3.1. Barrier Options.

3.1.1. *Down & Out Formula.* The payoff of a discrete down-and-out barrier (level D), also known as a point-barrier, European call option with strike K and monitoring times $\mathscr{T} = \{0, \Delta t, \ldots, N\Delta t = T\}$ is

$$v\left(R_N, \underline{R}_N\right) = S_0\left(e^{R_N} - e^k\right)^+ \chi_{[d,\infty)}\left(\underline{R}_N\right) \tag{3.3}$$

where $\underline{R}_N = \min_{t\in\mathscr{T}} R_t$, $k = \ln(K/S_0)$, $d = \ln(D/S_0)$. Here $\chi_A(x)$ denotes the characteristic function of $A \subseteq \mathbb{R}$, i.e., $\chi_A(x) = 1$ if $x \in A$, otherwise, $\chi_A(x) = 0$.

With a risk free rate r, the option price at time $t = 0$ is given by

$$C_D(S_0; K, D) = e^{-rT}\mathbb{E}_{\mathbb{Q}}\left[v\left(R_N, \underline{R}_N\right)\right] \tag{3.4a}$$

$$= e^{-rT}\int_{-\infty}^{\infty}\int_d^{\infty} v(\xi,\zeta)\mathbb{Q}\left(\{R_N \in [\xi, \xi + d\xi)\} \cap \{\underline{R}_N \in [\zeta, \zeta + d\zeta)\}\right) \tag{3.4b}$$

$$= e^{-rT}\int_{-\infty}^{\infty} w^{(\mathrm{d})}(\xi)\wp_N^{(\mathrm{d})}(\xi)d\xi \tag{3.4c}$$

where $\wp_N^{(\mathrm{d})}(\xi) = \mathbb{Q}\left(\{R_N \in [\xi, \xi + d\xi)\} \cap \{\underline{R}_{N-1} \geqslant d\}\right)/d\xi$ is the (defective) probability density of the random walk R_n at time N starting at $R_0 = 0$ and absorbed at boundary $d < 0$. The amended payoff is

$$w^{(\mathrm{d})}(R_N) = S_0 \left(e^{R_N} - e^k\right)^+ \chi_{[d,\infty)}(R_N). \tag{3.5}$$

Taking the Fourier z-transform, employing Parseval's relation and substituting the random walk result (2.17),

$$\begin{aligned}
C_D(S_0; K, D) &= \frac{\mathcal{Z}_N^{-1}}{2\pi} \int_{-\infty+iv}^{\infty+iv} \mathcal{F}^* w^{(\mathrm{d})}(\xi) \mathcal{Z}\mathcal{F}\wp_N^{(\mathrm{d})}(\xi) d\xi && \text{(3.6a)}\\
&= \frac{\mathcal{Z}_N^{-1}}{2\pi} \int_{-\infty+iv}^{\infty+iv} \hat{w}^{(\mathrm{d})}(-z) G_-(z) e^{idz} dz && \text{(3.6b)}\\
&= \frac{\mathcal{Z}_N^{-1}}{2\pi} \int_{-\infty+iv}^{\infty+iv} \hat{w}^{(\mathrm{d})}(-z) \left[\frac{1}{L(z)} - e^{idz} \frac{P_-(z) L_-(z)}{L(z)}\right] dz. && \text{(3.6c)}
\end{aligned}$$

The transformed payoff,

$$\begin{aligned}
\hat{w}^{(\mathrm{d})}(z) &= \int_d^\infty S_0 \left(e^x - e^k\right)^+ e^{izx} dx && \text{(3.7a)}\\
&= -S_0 \left[\frac{e^{(1+iz)\max(d,k)}}{1+iz} - \frac{e^{k+iz\max(d,k)}}{iz}\right], && \text{(3.7b)}
\end{aligned}$$

is valid for $\Im z > 1$.

Typically, for financial application, $K(z)$ is analytic for $-1 < \Im z < 0$. In the Gaussian case, since $K(z)$ is entire, an arbitrarily wide strip about the origin can be made free of the zeros of $L(z) = 1 - qK(z)$ by making $|q|$ sufficiently small. In such circumstances the integral in (3.8) is defined for any contour in the analytic strip with $v = \Im z < -1$.

Recognizing the first term integral in (3.6) as the vanilla option price $\tilde{C}_D = \tilde{C}_D(S_0; K, D)$ with payoff $w^{(\mathrm{d})}(R_N)$ gives

$$\boxed{C_D(S_0; K, D) = \tilde{C}_D - \frac{\mathcal{Z}_N^{-1}}{2\pi} \int_{-\infty+iv}^{\infty+iv} \hat{w}^{(\mathrm{d})}(-z) \left[e^{idz} \frac{P_-(z) L_-(z)}{L(z)}\right] dz,} \tag{3.8}$$

which is exactly analogous to the continuous time formula.

3.1.2. *Alternative Down & Out Formula When $D > K$.* For $D > K$ the condition that the price at expiry be above the barrier is redundant. In such situations it is useful to exploit the alternative representation of the killed probability densities obtained from (2.14a). This yields

$$C_D(S_0; K, D) = \frac{\mathcal{Z}_N^{-1}}{2\pi} \int_{-\infty+iv}^{\infty+iv} \hat{w}^{(\mathrm{d})}(-z) \left[e^{idz} \frac{P_+(z)}{L_+(z)}\right] dz. \tag{3.9}$$

Now observing the exponential terms in $\hat{w}^{(\mathrm{d})}(-z)$ cancel exactly with e^{idz}, since $d - \max(d, k) = 0$, the integration path may be deformed upwards where the integrand is analytic except at the poles of $\hat{w}^{(\mathrm{d})}(-z)$ at $z = -i$ and $z = 0$. Thus, by residue calculus

$$\boxed{C_D(S_0;K,D) = D\frac{P_+(-i)}{L_+(-i)} - K\frac{P_+(0)}{L_+(0)}.} \tag{3.10}$$

It is easy to verify that (3.10) may be re-expressed as

$$C_D(S_0;K,D) = S_0\mathbb{Q}^\star\left(\{\underline{S}_N > D\}\right) - K\mathbb{Q}\left(\{\underline{S}_N > D\}\right), \tag{3.11}$$

where $\mathbb{Q}^\star$ is a probability measure equivalent to the risk-neutral measure, $\mathbb{Q}$ corresponding to taking the stock price as numeraire. For the Gaussian this amounts to increasing the process drift by $\sigma^2/2$.

3.1.3. *Up & Out Formula.* The value of a discrete up-and-out barrier (level U) call option with strike K and monitoring times $\mathscr{T} = \{0, \Delta t, \ldots, N\Delta t = T\}$ is

$$\boxed{C_U(S_0;K,U) = \tilde{C}_U - \frac{\mathcal{Z}_N^{-1}}{2\pi}\int_{-\infty+iv}^{\infty+iv} \hat{w}^{(\mathrm{u})}(-z)\left(e^{iuz}\frac{Q_+(z)L_+(z)}{L(z)}\right)dz,} \tag{3.12}$$

where

$$\hat{w}^{(\mathrm{u})}(z) = S_0\left[\frac{e^{(1+iz)u}}{1+iz} - \frac{e^{k+izu}}{iz}\right] - S_0\left[\frac{e^{(1+iz)k}}{1+iz} - \frac{e^{k+izk}}{iz}\right] \quad (\Im z < 0), \tag{3.13}$$

$u = \ln(U/S_0) > 0$ and $\tilde{C}_U = \tilde{C}_U(S_0;K,U)$ is the corresponding vanilla price. The integration path must lie within the analytic strip with $v = \Im z > 0$.

An option which is knocked-in when a barrier is crossed may be calculated as the difference between corresponding knock-out and vanilla options.

3.2. **First-Touch Options.**

3.2.1. *Digitals.* A first-touch digital, equivalent to the rebate offered on many barrier options, pays £1 if and when S_t is observed below level D for the first time τ_d, before it expires worthless at time T. Discretely monitored, it has price

$$F_{FT}(S_0;D) = \mathbb{E}_{\mathbb{Q}}\left[e^{-r\tau_d}\right] \tag{3.14a}$$

$$= \sum_{n=1}^{N} e^{-rn}\mathbb{Q}\left(\{\tau_d = n\}\right). \tag{3.14b}$$

Using (2.26) and (2.29), $\mathbb{Q}\left(\{\tau_d = n\}\right) = G_-(0) = P_-(0)L_-(0)$, and so

$$\boxed{F_{FT}(S_0;D) = \sum_{n=1}^{N} e^{-rn}\mathcal{Z}_n^{-1}P_-(0)L_-(0).} \tag{3.15}$$

3.2.2. *Overshoot Option.* A claim paying the overshoot if and when S_t is observed below level D for the first time τ_d, before it expires worthless at time T, can be priced as

$$F_O(S_0;D) = \mathbb{E}_{\mathbb{Q}}\left[e^{-r\tau_d}\left(D-S_{\tau_d}\right)^+\right] \tag{3.16a}$$

$$= \sum_{n=1}^{N} e^{-rn}\int_{-\infty}^{d} S_0\left(e^d-e^x\right)^+ \wp_n^{(\mathrm{d})}(x)dx. \tag{3.16b}$$

Using the random walk result (2.30) for $\mathcal{F}\mathcal{Z}\wp_n^{(\mathrm{d})}(x) = e^{idz}G_-(z)$,

$$F_O(S_0;D) = -\sum_{n=1}^{N} De^{-rn}\mathcal{Z}_n^{-1}\int_{-\infty+iv}^{\infty+iv}\frac{P_-(z)L_-(z)}{z^2+iz}dz, \tag{3.17}$$

with $\Im v > 0$. Deforming the integration contour downwards yields

$$\boxed{F_O(S_0;D) = \sum_{n=1}^{N} De^{-rn}\mathcal{Z}_n^{-1}\left[P_-(-i)L_-(-i)-P_-(0)L_-(0)\right].} \tag{3.18}$$

The corresponding option in the Black-Scholes model with continuous monitoring is always worthless due to the Brownian motion having continuous trajectories.

3.3. **Hindsight Options.** A hindsight call option, also known as a look-back, with strike K has payoff $\left(\bar{S}_T-K\right)^+$. Monitored discretely, $w^{(\mathrm{h})}\left(\bar{R}_N\right) = \left(S_0e^{\bar{R}_N}-K\right)^+$ has Fourier transform

$$\hat{w}^{(\mathrm{h})}(z) = -K\frac{e^{izk}}{z^2-iz}, \tag{3.19}$$

valid for $\Im z > 1$, where $k = \ln(K/S_0)$. Thus, the hindsight price is

$$C_H(S_0;K) = e^{-rT}\int_{-\infty+iv}^{\infty+iv} w^{(\mathrm{h})}(x)\wp_n^{(\mathrm{m})}(x)dz. \tag{3.20}$$

The Fourier z-transform of the probability density of $\bar{R}_n$ is given by $\mathcal{F}\mathcal{Z}\wp_n^{(\mathrm{m})}(x) = H_+(z)$, (2.40), and so

$$\boxed{C_H(S_0;K) = -\frac{e^{-rT}\mathcal{Z}_N^{-1}}{2\pi}\int_{-\infty+iv}^{\infty+iv} K\frac{e^{-izk}}{z^2+iz}\left[\frac{1}{L_-(0)L_+(z)}\right]dz,} \tag{3.21}$$

where $v = \Im z < -1$.

In the special case $S_0 > K$, i.e. $k < 0$, the integration contour may be deformed upwards to pick up the residues at $z = -i$ and $z = 0$, so that

$$C_H(S_0;K) = e^{-rT}\mathcal{Z}_N^{-1}\left[\frac{S_0}{L_-(0)L_+(-i)}\right] - K. \tag{3.22}$$

3.4. **Double-Barrier Options.** A double-barrier call option is equivalent to a vanilla call option, but rendered worthless if on any monitoring date S_t is observed above an upper barrier U or below a lower barrier D. It has price

$$C_{DB}(S_0;K,D,U) = e^{-rT}\int_{-\infty}^{\infty} w^{(\mathrm{ud})}(x)\wp_N^{(\mathrm{ud})}(x)dx, \tag{3.23}$$

where the payoff $w^{(\mathrm{ud})}(x)$ is written as

$$w^{(\mathrm{ud})}(x) = S_0 \left(e^x - e^k\right)^+ \chi_{[d,u]}(x). \tag{3.24}$$

Using (2.59) and identifying a term $\tilde{C}_{DB} = \tilde{C}_{DB}(S_0; K, D, U)$ equal to the vanilla price with payoff $w^{(\mathrm{ud})}(x)$,

$$\boxed{C_{DB} = \tilde{C}_{DB} - \frac{e^{-rT}\mathcal{Z}_N^{-1}}{2\pi}\int_{-\infty+iv}^{\infty+iv} \hat{w}^{(\mathrm{ud})}(-z)\left[\frac{e^{idz}J_-(z)}{L(z)} + \frac{e^{iuz}J_+(z)}{L(z)}\right]dz.} \tag{3.25}$$

where the transformed payoff

$$\hat{w}^{(\mathrm{ud})}(z) = S_0\left[\frac{e^{(1+iz)u}}{1+iz} - \frac{e^{k+izu}}{iz}\right] - S_0\left[\frac{e^{(1+iz)\max(d,k)}}{1+iz} - \frac{e^{k+iz\max(d,k)}}{iz}\right] \tag{3.26}$$

is entire, so the restriction on the integration path is simply to be within the analytic strip.

3.5. **Double First-Touch Options.** A double first-touch pays out if and when S_t is observed outside the finite interval $[D, U]$ for the first time, say £A above U, £B below D, before it expires worthless at time T. Monitored discretely it can be priced using (2.61):

$$\boxed{F_{DFT}(S_0; U, D) = \sum_{n=1}^{N} e^{-rn}\mathcal{Z}_n^{-1}\left[AJ_+(0) + BJ_-(0)\right].} \tag{3.27}$$

4. Asymptotic Analysis

Discretely monitored results are now considered as monitoring frequency increases, $\Delta t \to 0$. This provides continuous time formulae and is important in the derivation of continuity correction approximations.

4.1. **Continuum Limits.** Firstly, without limiting the discussion to any particular Lévy process, deduction of continuous-time formulae is presented.

4.1.1. *Relationship Between z-Transform and Laplace Transform.* Expressing the Laplace transform as its Riemann sum limit,

$$\mathcal{L}f(t) = \int_0^\infty f(t)e^{-\lambda t}dt \tag{4.1a}$$

$$= \lim_{\Delta t \to 0}\sum_{n=0}^{\infty}\left(t_{n+1} - t_n\right)e^{-\lambda t_n}f\left(t_n\right) \tag{4.1b}$$

$$= \lim_{\Delta t \to 0}\Delta t\sum_{n=0}^{\infty}q^n f_n \tag{4.1c}$$

$$= \lim_{\Delta t \to 0}\Delta t\mathcal{Z}f_n, \tag{4.1d}$$

where increments $t_{n+1} - t_n = \Delta t$ are fixed, $t_n = n\Delta t$, $q = e^{-\lambda\Delta t}$ and $f_n = f\left(n\Delta t\right)$.

4.1.2. *Limit of Wiener-Hopf Factors $L_\pm(z)$ as $\Delta t \to 0$.* The Lévy characteristic function on an interval Δt is generally of the form

$$K(z) = \exp\{-\psi(z)\Delta t\}, \tag{4.2}$$

where $\psi(z)$ is known as the *characteristic exponent.* Setting $q = e^{-\lambda\Delta t}$, the limit $l(z)$ is defined as

$$\begin{aligned} l(z) &= \lim_{\Delta t\to 0}\left[L(z)/\,\Delta t\right] && \text{(4.3a)}\\ &= \lim_{\Delta t\to 0}\left[\left(1 - e^{-(\lambda+\psi(z))\Delta t}\right)\Big/\,\Delta t\right] && \text{(4.3b)}\\ &= \lambda + \psi(z). && \text{(4.3c)} \end{aligned}$$

The limit Wiener-Hopf factors

$$l_\pm(z) = \lim_{\Delta t\to 0}\left[L_\pm(z)/\sqrt{\Delta t}\right] \tag{4.4}$$

must then satisfy

$$l_-(z)l_+(z) = \lambda + \psi(z). \tag{4.5}$$

Provided $\lambda+\psi(z)$ presents an analytic strip free from zeros and the integrals defining $l_\pm(z)$ converge, the factorization proceeds in a similar manner to the discrete case.

4.1.3. *Continuous-Time Formulae.* The continuum limits for the results given in Section 2 and Section 3 are realized by substituting $l(z)$ for $L(z)$ (and so $l_\pm(z)$ for $L_\pm(z)$) and taking the inverse Laplace transform $\mathcal{L}_t^{-1}$ instead of the inverse z-transform $\mathcal{Z}_n^{-1}$. Also, for continuous-time first-touch options, $\sum_{n=0}^{N} \mathcal{Z}_n^{-1} f(q)$ is interpreted as $\int_0^T \mathcal{L}_t^{-1} f(\lambda)dt$.

Expression (4.5) in combination with (2.41) give the factorization implicit in Baxter-Donsker formulae (Baxter and Donsker (1957)), while with (2.30) relates to Pecherskii-Rogozin formulae (Pecherskii and Rogozin (1969)).

A formula for the price of continuous-time barrier options under jump-diffusion processes in the Laplace transform domain is discussed in Lewis (2003), see also Boyarchenko and Levendorskii (2002). For the continuous-time double-barrier options pricing problem, Pelsser (2000) and Hui et al. (2000) take an analytic approach, while Geman and Yor (1996) a more probabilistic one to provide a solution in the Gaussian (Black-Scholes) case using the Laplace transform. Both approaches rely on the provision of transformed probability densities (for example, a Fourier series) corresponding to the continuum limit of the coupled integral equations (2.57) and

(2.58) for Brownian motion. Recently Grudsky (2007) provided a formal solution to the continuousely monitored double-barrier problem by solving the continuum limit of the modified Wiener-Hopf equation (2.50) using Toeplitz operator theory.

4.2. **Continuity Corrections.** Now, focusing on Gaussian processes, further examination of asymptotic behaviour relates results to adjusted continuous-time formulae which better approximates discrete-time values by the so called 'continuity correction' approach.

4.2.1. *Behaviour of Wiener-Hopf Factors $L_{\pm}(z)$ as $\Delta t \to 0$.* Consider the random walk with normally distributed increments $\mathscr{N}[0, \Delta t]$, making the characteristic exponent $\psi(z) = z^2/2$. The factorization components of

$$M(z) = M(z, \lambda) = 1 - e^{-(\lambda + \psi(z))\Delta t} \tag{4.6}$$

are given by

$$\ln\left(M_{\pm}(z)\right) = \pm \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\ln\left(1 - e^{-(\lambda+\psi(\xi))\Delta t}\right)}{\xi - z} d\xi \quad (\Im\xi \lessgtr \Im z)\,. \tag{4.7}$$

Notice, by symmetry, $M_{-}(z) = M_{+}(-z)$. Separating out the logarithmic singularity as $\Delta t \to 0$ gives

$$\pm \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\ln\left(\left(\lambda + \psi(\xi)\right)\Delta t\right)}{\xi - z} d\xi = \ln\left(m_{\pm}(z)\right) + \frac{1}{2}\ln(\Delta t), \tag{4.8}$$

where

$$\ln\left(m_{\pm}(z)\right) = \pm \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\ln\left(\lambda + \psi(\xi)\right)}{\xi - z} d\xi \quad (\Im\xi \lessgtr \Im z)\,. \tag{4.9}$$

It remains to examine the behaviour of the integral

$$\kappa(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \ln\left[\frac{1 - e^{-(\lambda + \xi^2/2)\Delta t}}{(\lambda + \xi^2/2)\Delta t}\right] \frac{d\xi}{(\xi - z)}. \tag{4.10}$$

By symmetry of the integrand,

$$\kappa(z) = 2z \int_{0}^{\infty} \ln\left[\frac{1 - e^{-(\lambda + \xi^2/2)\Delta t}}{(\lambda + \xi^2/2)\Delta t}\right] \frac{d\xi}{(\xi^2 - z^2)}, \tag{4.11}$$

then making the substitution $\eta^2 = \xi^2 \Delta t$,

$$\kappa(z) = -\frac{iz\sqrt{\Delta t}}{\pi} \int_{0}^{\infty} \ln\left[\frac{1 - e^{-(\lambda\Delta t + \eta^2/2)}}{(\lambda\Delta t + \eta^2/2)}\right] \frac{d\eta}{(\eta^2 - z^2\Delta t)} \tag{4.12a}$$

$$= -\frac{iz\sqrt{\Delta t}}{\pi} \int_{0}^{\infty} \ln\left[\frac{1 - e^{-\eta^2/2}}{(\eta^2/2)}\right] \frac{d\eta}{\eta^2} + \mathcal{O}\left(\Delta t^{\alpha}\right) \tag{4.12b}$$

$$= iz\beta\sqrt{\Delta t} + \mathcal{O}\left(\Delta t^{\alpha}\right), \tag{4.12c}$$

where $\beta = -\zeta\left(1/2\right)/\sqrt{2\pi} \approx 0.5826$ (with $\zeta\left(x\right)$ the Riemann-Zeta function) and $\alpha \geqslant 1$ to agree with the Taylor expansion of (4.6).

Recombining (4.8) with (4.12) and exponentiating gives

$$M_{\pm}(z)/\sqrt{\Delta t} = m_{\pm}(z)e^{\pm iz\beta\sqrt{\Delta t}} + \mathcal{O}\left(\Delta t\right). \tag{4.13}$$

This is related to the random walk with normally distributed increments $\mathscr{N}\left[\mu\Delta t, \sigma^2\Delta t\right]$, whose characteristic exponent is $\psi(z) = -i\mu z + \sigma^2 z^2/2$, by completion of the square,

$$L(z,\lambda) = M\left(\sigma z - i\mu/\sigma, \lambda + \mu^2/2\sigma^2\right), \tag{4.14}$$

that is

$$L_{\pm}(z)/\sqrt{\Delta t} = l_{\pm}(z)e^{\pm i(\sigma z - i\mu/\sigma)\beta\sqrt{\Delta t}} + \mathcal{O}\left(\Delta t\right). \tag{4.15}$$

4.2.2. *Approximate Formulae.* Approximations for killed distributions and first crossing probabilities, and so single/double barrier and first touch option prices, are obtained by observing that the Wiener-Hopf factors always combine as $e^{idz}L_{-}(z)/e^{id\xi}L_{-}(\xi)$ or $e^{iuz}L_{+}(z)/e^{iu\xi}L_{+}(\xi)$ (see, for example, (2.24) in (2.23)). The constant coefficients of $\sqrt{\Delta t}$ clearly cancel. Thus, replacing d with $d + \sigma\beta\sqrt{\Delta t}$ and/or u with $u - \sigma\beta\sqrt{\Delta t}$ entirely eliminates the $\sqrt{\Delta t}$ behaviour. The explanation of this phenomenon is that displacing the discretely monitored barrier towards the diffusion proportionately to $\sqrt{\Delta t}$ compensates for price paths crossing barriers between monitorings.

This implies that displacing the continuous-time barriers to $De^{-\sigma\beta\sqrt{\Delta t}}$ and $Ue^{\sigma\beta\sqrt{\Delta t}}$ provides an approximation to discrete-time probability densities with an $\mathcal{O}\left(\Delta t\right)$ error. The same level approximation is achieved for option prices where the terminal payoff does not explicitly involve the barrier position. If, however, the payoff does involve the barrier position, barrier movement spuriously re-introduces a small $\mathcal{O}\left(\sqrt{\Delta t}\right)$ error.

The hindsight option also admits an approximation, since replacing k with $k + \sigma\beta\sqrt{\Delta t}$ eliminates the $\sqrt{\Delta t}$ behaviour of the discrete-time results as $\Delta t \to 0$. The implied continuous-time approximation to discrete-time prices which is achieved by substituting $S_0 e^{\sigma\beta\sqrt{\Delta t}}$ for S_0 in the continuous-time formula has $\mathcal{O}\left(\Delta t\right)$ error.

Development of these approximations is described in Broadie et al. (1999), Howison and Steinberg (2006) and references therein, where they display good accuracy for frequent monitoring and a configuration where there is substantial separation between barrier and strike. The connection with less direct derivations of the continuity correction is given as a supplementary result in Appendix A2, calculating the asymptotic behaviour of the expected maximum of a Gaussian random walk.

5. Numerical Results

In this section a range of numerical results are offered based on the formulae derived in Section 3. The Black-Scholes model is assumed for all results presented herein.

5.1. **Comparison of Computational Techniques.** The results of experiments pricing down-out barrier options using Mathematica for the techniques presented below (in addition to continuous-time prices, prices obtained by continuity correction and simulated prices) are presented in Table I. Common to all numerical procedures is an efficient method for the inverse z-transform. This is discussed first.

Barrier Level	Numerical Integration	Truncated Summation	Padé Approximate	Continuous Price	Continuity Correction	Trajectory Simulation
90.0	6.24292	6.24292	6.24292	5.97724	6.24813	6.24000
95.0	5.67111	5.67110	5.67111	4.39750	5.64562	5.66326
99.0	4.48917	4.31896	4.48917	1.17079	4.04952	4.50257
99.5	4.29702	4.32696	4.29702	0.60480	3.73716	4.30592
99.9	4.13824	6.67852	4.13824	0.12408	3.46577	4.14153
Time (s)	1800	120	0.2	<0.01	<0.01	60

Table I: Comparison of Down-Out Barrier Option Prices Calculated by Different Techniques†

† Process and payoff parameters are set to $N = 5$, $S_0 = K = 100$, $r = 0.1$, $\sigma = 0.3$ and $T = 0.2$. The z-transform parameter size is set using $\gamma = 5$. The summation involves 200×100 terms, but, as in Fusai et al. (2006), $L_{\pm}(z)$ is calculated by numerical integration, this being more efficient than using the exact expression. The Padé approximation of $K(z)$ used is [24/36]. Simulations sample 10^6 price trajectories.

5.1.1. *Inverse z-Transform.* Choosing a circular (radius $\rho < 1$) integration contour, the inverse z-transform (2.20) expressed in polar coordinates is

$$\mathcal{Z}_N^{-1} f(q) = \frac{1}{2\pi\rho^N} \int_0^{2\pi} f(\rho e^{i\theta}) e^{-iN\theta} d\theta. \tag{5.1}$$

This may be evaluated by numerical integration (using e.g. Mathematica routine `NIntegrate[]`), but for large N this becomes extremely slow.

Abate and Whitt (1992) propose an alternative method based on approximating (5.1) using the trapezoidal rule with step size π/N:

$$\mathcal{Z}_N^{-1} f(q) \approx \frac{1}{2\pi\rho^N} \left\{ f(\rho) + (-1)^N f(-\rho) + 2\sum_{j=1}^{N-1} (-1)^j \Re\left[f\left(\rho e^{j\pi i/N}\right)\right] \right\}. \tag{5.2}$$

This has an error bound $\rho^{2N}/(1-\rho^{2N})$, so practically, to have an accuracy of at least $10^{-\gamma}$ requires $\rho = 10^{-\gamma/2N}$. It is important to adjust ρ in this way so calculation of $\rho^N = 10^{-\gamma/2}$ in the denominator of (5.2) does not introduce error.

The evaluation of first touch and overshoot options involve sums of inverse z-transforms which are efficiently calculated by observing $\mathcal{Z}_n^{-1} f(q) = \mathcal{Z}_N^{-1} q^{(N-n)} f(q)$, that is, only a single inversion is required.

5.1.2. *Numerical Integration.* All integrals can be computed numerically. For the down-out barrier example, the convergence of (3.8) and in turn (2.15) are improved by splitting integrands using $1/L(z) = qK(z)/L(z) + 1$ and observing that the parts involving the second term vanish due to analyticity. The explicit q^2 factor introduced (due to nested integration) also allow a reduction by 2 of the number of terms necessary to approximate the inverse z-transform. Yet, for practical purposes numerical integration remains too slow to be effective.

5.1.3. *Truncated Summation.* For the Gaussian case, $1/L(z)$ is meromorphic and so option prices may be evaluated using residue calculus. This gives rise to a solution in terms of infinite multiple sums as demonstrated for the down-out barrier option in Appendix A3. For this option, when $d << k$ (i.e. $D << K$), the exact sums converge rapidly due to the exponential terms. These must be truncated to obtain numerical values, so as $d \to k$ and convergence becomes slow, maintenance of accuracy becomes time consuming.

The up-out and double barrier call options, with an extra discontinuity in their payoffs, are additionally challenging for most numerical and approximate pricing techniques. Clearly, in the pricing formulae (3.12) and (3.25) the exponential terms associated with the upper barrier exactly cancel. For summation type solutions, this manifests itself in slow convergence even when $u >> 0$. To achieve accurate prices truncated sums must retain many terms, which renders the technique slow and impractical.

5.1.4. *Padé Approximation.* Accurate results are achieved most efficiently by obtaining a symmetric rational approximation to $L(z)$, which is the quotient of two polynomials of equal degree, say M. This may be realized by a Padé $[L, M]$ approximation to $K(z)$ (using e.g. Mathematica routine `Pade[K(z),{z,0,L,M}]`). If it exists, a Padé approximant $[L, M]$ is uniquely determined from the Taylor expansion of $K(z)$ about any given point. The approximate factorization of $L(z)$ into non-zero analytic functions in $\Im z \gtrless 0$ from $1 - q\,[L, M]$ follows by inspection after polynomial factorization (using e.g. Mathematica routine `NSolve[]`). For particulars on the approximation of Wiener-Hopf kernels by Padé approximants the reader is referred to Abrahams (2000).

Subsequent calculations involve summation of a small number of residues algebraically calculated, as in the exact summation case where residues are calculated by numerical integration at a truncated infinite set of exact poles; now, using the approximation, the residues at a relatively small finite set of artificial poles are immediately available. An additional step in the double barrier case is to solve (2.57) and (2.58). Deforming the integration contours to pick up the residues at the artificial poles results in implicit linear approximations to $J_{\pm}(z)$, i.e. expressed as a linear combination of $J_{\mp}(z)$ evaluated at the artificial poles, which may be solved over the set of artificial poles by simple linear algebra. The procedure is outlined in Appendix A5.

The use of Mathematica routines `Pade[]` and `NSolve[]` constitute an efficient calculation of prices to around 4-5 decimal places, but root-finding algorithms are notorious for being computationally burdensome and lack precision with high degree polynomials. Furthermore, the optimized algorithms implemented by Mathematica are not easily ported to compiled programming languages such as C++ or Java. This drawback is overcome (at least in the Gaussian case) by using the algorithm described in Appendix A4, where rational approximations to $L(z) = 1 - qK(z)$ are obtained from saved list of zeros and poles of an approximation to $1 - e^{-z}$ (which may previously be calculated using high-precision arithmetic) for all process parameters and values of q. Requiring neither numerical integration nor numerical solutions to equations at run-time, this approach is, obviously, very efficient and trivial to implement in, for example, C++.

In addition to the computational advantage gained using rational approximations, errors in prices calculated by the method described are insensitive to the configuration of barrier levels and strike prices. This is thanks to prices being calculated *exactly* in terms of approximated characteristic functions, which depend on the process alone. Thus, the method does not rely in the convergence of integrals or sums parameterized by option payoff parameters.

5.2. **Benchmarking Calculated Option Prices.** The comparison of prices calculated by the Padé approximation method (using [24/36] and [40/60] approximants to $K(z)$ which ensure rapid decay at infinity) with other results in the literature is based on C++ implementation of the procedure described in Appendix A4. The z-transform parameter size is set using $\gamma = 5.75$. Table II gives calculated prices for down-out barrier options compared with those found in Broadie and Yamamoto (2005). Table III compares calculated double-barrier option prices with those in Fusai and Recchioni (2007). A search in the option pricing literature for discretely

Number of Monitorings	Padé $[24, 36]$ Approximation†	Padé $[40, 60]$ Approximation	B&Y Result
5	4.4891749110	4.4891724312	4.4891724312
25	2.8124392593	2.8124392982	2.8124392982
50	2.3363869060	2.3363868958	2.3363868958

Table II: Calculated Down-Out Barrier Option Prices Compared With Broadie & Yamamoto (B&Y)*

* Process and payoff parameters are set to $D = 99$, $S_0 = K = 100$, $r = 0.1$, $\sigma = 0.3$ and $T = 0.2$.
† Computational times less than 0.02 seconds.

Number Monitorings	Padé $[24, 36]$ Approximation†	Padé $[40, 60]$ Approximation	F&R Result
50	0.1639413201	0.1639410637	0.163941
100	0.1189380004	0.1189381452	0.118938
150	0.1016929798	0.1016929046	0.101692

Table III: Calculated Double Barrier Option Prices Compared With Fusai & Recchioni (F&R)*

* Process and payoff parameters are set to $D = 90$, $U = 110$ $S_0 = 100$, $K = 95$, $r = 0.05$, $\sigma = 0.1$ and $T = 1$.
† Average computational time circa 10 seconds.

monitored first-touch options (or barrier rebate) failed to provide any price data with which the Padé approximation method could be compared. Consequently, calculated first overshoot option prices rely on the simulation of 10^8 price trajectories using Mersenne twister random number generation with antithetic variates for comparison in Table IV.

The computational cost for the Padé approximation method increases at most linearly with monitoring frequency. Single barrier calculations in this section complete within a fraction of a second, while double barrier calculations are slightly slower because of the multiple matrix inversions required. Calculations using Padé

Number of Monitorings	Padé $[24, 36]$ Approximation†	Padé $[40, 60]$ Approximation	Simulated Result†
5	2.7069262554	2.7069260783	2.7069±0.0003
25	1.3762929854	1.3762930537	1.3764±0.0002
50	0.9830350268	0.9830348953	0.9829±0.0001

Table IV: Calculated Overshoot Option Prices Compared With Simulated Results*

* Process and payoff parameters are set to $D = 99$, $S_0 = K = 100$, $r = 0.1$, $\sigma = 0.3$ and $T = 0.2$.
† Computational times less than 0.5 seconds. Simulations circa 5 minutes.

[24/36] usually give around six-seven digits accuracy after the decimal point in approximately a third the computational time of Padé [40/60], which usually give an accuracy of around ten decimal places. With model error and the difficulty inherent in parameter estimation, such a level of accuracy is only really of theoretical interest.

5.3. **Frequent Monitoring Behaviour.** No exact investigation of relatively frequent monitoring has been found in the literature. With only a linear increase in computational burden it is possible to calculate the exact price for discretely monitored options with good precision for up to a million monitoring times, approximately every second of the working day. In some cases the convergence of discrete prices to continuous prices is very slow, with significant discrepancy even with what may have been considered sufficient for the continuity assumption. Table V gives the price of discretely monitored down-out options and compares these to the prices obtained with the continuity correction approximation.

Number of Monitorings	Padé $[24, 36]$ Approximation†	Padé $[40, 60]$ Approximation	Continuity Correction
10	3.6728067772	3.6728077261	3.3813685612
100	1.9905218699	1.9905218655	1.9685666052
1000	1.4334240504	1.4334240496	1.4334835904
10000	1.2549191300	1.2549191298	1.2549196172
100000	1.1975021599	1.1975021598	1.1975023139
1000000	1.1792498404	1.1792498404	1.1792498768
∞	1.1707930349	1.1707930349	1.1707930349

Table V: Calculated Down-Out Option Prices Compared With Results Obtained By Continuity Correction†

† Process and payoff parameters are set to $D = 99$, $S_0 = K = 100$, $r = 0.1$, $\sigma = 0.3$ and $T = 0.2$.
† Computational times are less than 0.01, 0.03, 0.33, 3.3, 33 and 330 seconds.

6. Conclusion

The substantial difference between the prices of options where the underlying asset is discretely monitored and those calculated assuming a continuous model are well known. This work provided various probabilistic identities, some new, others very famous, but cast here in an analytically convenient form which gives a simple and coherent framework for the calculations of discretely monitored path dependent option prices. Furthermore, not only are discrete-time pricing formulae easily related to those in continuous-time, but the discrete-time prices are explicitly shown to be asymptotically closer to continuous-time results with adjusted parameters. This work also examined the numerical evaluation of prices. Padé approximants provide for fairly effortless calculations which exhibit good accuracy and efficiency. Results for barrier and hindsight options compare well with numerical techniques avaliable in the literature. Discretely monitored double barrier and first touch options are priced by analytic methods for the first time, but agree well with prices obtained by simulation. It is also worth noting that at monitoring times the hedging factors $\Delta = \frac{\partial C}{\partial S}$ and $\Gamma = \frac{\partial^2 C}{\partial S^2}$ (for contract priced $C(S)$) may be obtained directly by differentiation at no extra computational cost.

In addition to standard traded contracts, a novel option based on the overshoot of a barrier between monitoring times was introduced and priced. Although not common, this contract would have obvious practical utility in insuring against large intermonitoring losses. Moreover, it provides insight into the intrinsically discrete nature with which assets underlying derivative contracts are monitored.

The convergence of prices to their continuous-time limit is in some situations extremely slow. Even for what may be considered continuous monitoring (e.g. every second), if the discrete nature is not accounted for, significant misprising can ensue. The plausibility of ever monitoring at such high frequencies, that markets actually operate in a continuous-time framework is highly questionable. Certainly, the discrete nature of monitoring is a necessary consideration for all path dependent options. Work is currently ongoing to perform a rigorous asymptotic analysis to provide higher order approximates to prices when the intermonitoring time is not small enough for Broadie's continuity correction approximation to be sufficiently accurate.

The distribution of market prices are known to deviate from the Black-Scholes model in a number of ways, e.g. 'fat tails' and skewness. Models with jumps (or Lévy processes) are one attempt to simply improve market fit. The price of discretely monitored options contingent upon an asset driven by a Lévy process fit into the pricing framework described here with substitution of the appropriate characteristic function, which is generally available, though in some cases there are technical issues concerning their rational approximation. A comparison of prices under different models is in progress. The method is also currently being extended to price a variety of more esoteric contracts.

APPENDICES

A1. **Formulae for Vanilla Option Prices in Fourier Space.** Using the notation in the main text, $\wp_N(x)$ is the probability density of $R_T = \ln(S_T/S_0)$, $T = N\Delta t$. The payoff function for an European call is $w^{(c)}(x) = S_0\left(e^x - e^k\right)^+$ and for a put $w^{(p)}(x) = S_0\left(e^k - e^x\right)^+$, where $k = \ln(K/S_0)$. Employing Parseval's identity, the value of the call option is

$$C_V(S_0;K) = \frac{e^{-rT}}{2\pi}\int_{-\infty+iv}^{\infty+iv}\hat{w}^{(c)}(-z)\phi(z)dz, \tag{A1.1}$$

where $\hat{w}^{(c)}(z) = \mathcal{F}w^{(c)}(x)$ is the Fourier transform of the call payoff,

$$\hat{w}^{(c)}(z) = -S_0\frac{e^{(1+iz)k}}{z(z-i)}, \tag{A1.2}$$

valid for $\Im z > 1$, and $\phi(z) = \mathcal{F}\wp_N(x) = \mathbb{E}\left[e^{izR_T}\right]$ is the Fourier transform of the distribution. The integration path $v = \Im z$ must lie where both $\hat{w}^{(c)}(-z)$ and $\phi(z)$ are well defined. For the (Gaussian) Black-Scholes model, $\phi(z)$ is entire, so any $v = \Im z < -1$ suffices.

The vanilla put has the same transform $\hat{w}^{(p)}(z) = \hat{w}^{(c)}(z)$, but different region of validity $\Im z < 0$.

Normalization requires $\phi(0) = 1$ and the martingale property requires $\phi(-i) = e^{rT}$, so the residue of the integrand at its simple poles $z = 0$ and $z = -i$ are $-\frac{Ke^{rT}}{2\pi i}$ and $\frac{S_0}{2\pi i}$ respectively. Moving the integration contour in (A1.1) to $v = \Im z > 0$ picks up the residues to give $C_V(S_0;K) = P_V(S_0;K) + S_0 - Ke^{-r(T-t)}$, where P_V is the value of the vanilla put, i.e. put-call parity is recovered.

Using partial fractions the call price can be cast in Black-Scholes form,

$$\begin{aligned} C_V(S_0;K) &= \frac{Ke^{-rT}}{2\pi i}\int_{-\infty+iv}^{\infty+iv} e^{-ikz}\phi(-z)\left(\frac{1}{z-i} - \frac{1}{z}\right)dz && \text{(A1.3a)} \\ &= S_0\Pi_1 - Ke^{-rT}\Pi_2. && \text{(A1.3b)} \end{aligned}$$

The integral Π_1 can be interpreted as the option delta $\Pi_1 = \Delta = \frac{\partial C_V}{\partial S_0}$ (since $\frac{\partial \Pi_1}{\partial k} = e^k e^{-rT}\frac{\partial \Pi_2}{\partial k}$) and Π_2 as the probability that the option expires in the money $\Pi_2 = \mathbb{Q}(X_T > -k) = \mathbb{Q}(S_T > K)$.

This formalism for other common vanilla options and various popular Lévy models is discussed in Lewis (2001). The work of Bakshi and Madan (2000) is also relevant.

A2. **Asymptotic Expansion of Gaussian Random Walk's Expected Maximum.** Consider the driftless Gaussian random walk R_n with increments distributed as $\mathscr{N}\left[0, \sigma^2 \Delta t\right]$. The Fourier z-transform of the density of the maximum $\bar{R}_n$ is $H_+(z)$, given by (2.40). Recognising the relationship between the z-transform and Laplace transform given by (4.1), the behaviour of the expected maximum as $\Delta t \to 0$ is given by

$$\text{(A2.1a)} \qquad \mathbb{E}\left[\bar{R}_n\right] = \mathcal{L}_t^{-1}\left[-i\Delta t \frac{\partial}{\partial z} H_+(z)\right]_{z=0}$$

$$\text{(A2.1b)} \qquad = \mathcal{L}_t^{-1}\left[-i\Delta t \frac{L'_+(0)}{L(0)L_+(0)}\right].$$

From the representation of $L_+(z)$ given by (4.15),

$$\text{(A2.2a)} \qquad i\Delta t \frac{L'_+(0)}{L(0)L_+(0)} = \frac{1}{\lambda}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\ln\left(\lambda + \sigma^2\eta^2/2\right)}{\eta^2} d\eta - \beta\sigma\sqrt{\Delta t}\right]$$

$$\text{(A2.2b)} \qquad = \frac{\sigma}{\sqrt{2}\lambda^{2/3}} - \frac{\beta\sigma\sqrt{\Delta t}}{\lambda} + \mathcal{O}\left(\Delta t\right).$$

Hence, taking the inverse transform,

$$\text{(A2.3)} \qquad \mathbb{E}\left[\bar{R}_n\right] = \sigma\sqrt{\frac{2t}{\pi}} - \beta\sigma\sqrt{\Delta t} + \mathcal{O}\left(\Delta t\right).$$

The first term is the expected maximum of a Brownian motion, the second becomes the continuity correction.

A3. **Exact Solution by Summation of Residues.** The Gaussian characteristic function given by (3.2) is entire, so the integrand of the integral in (3.8) is analytic in the lower-half plane ($\Im z < -1$) except at the zeros of $L(z) = 1 - qK(z)$. It also vanishes as $|z| \to \infty$. Deforming the integration contour downwards gives

$$\text{(A3.1)} \qquad C_D(S_0; K, D) = \tilde{C}_D + i\mathcal{Z}_N^{-1} \sum_{n=-\infty}^{\infty} \left(P_-(\gamma_n^-) L_-(\gamma_n^-) R\left(\gamma_n^-\right) \hat{w}^{(\mathrm{d})}\left(-\gamma_n^-\right) e^{id\gamma_n^-}\right),$$

where the sum is over all the poles γ_n^- of $1/L(z)$ in $\Im z < 0$, $R(\gamma_n^-)$ being their residues. Similarly, from (2.15),

$$\text{(A3.2)} \qquad P_-(z) = \sum_{n=-\infty}^{\infty} \frac{e^{-id\gamma_n^+} L_+\left(\gamma_n^+\right) R\left(\gamma_n^+\right)}{\gamma_n^+ - z}.$$

The solutions of $L(\gamma_n) = 1 - qK(\gamma_n) = 0$ are given by

$$\text{(A3.3)} \qquad \gamma_n^{\pm} = \frac{i\left(r - \sigma^2\Delta t/2\right)}{\sigma^2 \Delta t} \pm \frac{1}{\sigma\sqrt{\Delta t}} \sqrt{2\ln q + 4n\pi i - \frac{\left(r - \sigma^2\Delta t/2\right)^2}{\sigma^2\Delta t}} \quad (n \in \mathbb{Z}).$$

The residue $R(\gamma_n^\pm)$ at the simple pole $\gamma_n^\pm$ of $1/L(z)$ is calculated as

$$(A3.4) \qquad R\left(\gamma_n^\pm\right) = \lim_{z\to\gamma_n} \frac{z-\gamma_n^\pm}{L(z)-L\left(\gamma_n^\pm\right)} = \frac{1}{L'\left(\gamma_n^\pm\right)} = \frac{1}{\gamma_n^\pm\sigma^2\Delta t - i\left(r-\sigma^2\Delta t/2\right)},$$

where $qe^{i\gamma_n^\pm\left(r-\sigma^2\Delta t/2\right)-\left(\gamma_n^\pm\right)^2\sigma^2\Delta t/2} = 1$ has been used.

An exact representation can also be given for $L_\pm(z)$ by writting

$$(A3.5) \qquad L(q,z) = M\left(q\exp\left\{-\frac{1}{2}\left(\frac{r}{\sigma}-\frac{\sigma}{2}\right)^2\Delta t\right\}, \sigma\sqrt{\frac{\Delta t}{2}}z - i\left(\frac{r}{\sigma}-\frac{\sigma}{2}\right)\sqrt{\frac{\Delta t}{2}}\right),$$

where

$$(A3.6) \qquad M(q,z) = 1-qe^{-z^2},$$

whose factors can be expressed as

$$(A3.7) \qquad M_\pm(z) = \exp\left\{\frac{1}{2}\sum_{n=1}^{\infty}\frac{q^n e^{-nz^2}}{n}\left(1\pm\operatorname{erf}\left(i\sqrt{n}z\right)\right)\right\}.$$

This representation of the Wiener-Hopf factors can be obtained most directly from Spitzer's identity (2.44).

A4. **Rational Approximation of $L(z) = 1 - qK(z)$.** The following is a two part procedure for calculating rational approximations. The first part of the procedure finds the Padé approximation to e^{-z}, then the zeros and poles of the resulting approximation to $1-e^{-z}$. This may demand high precision arithmetic, but it need only be completed once for a given choice of Padé number. An approximation to $L(z)$ is then obtained by tracing all its zero and poles from those of $1-e^{-z}$.

The Padé approximation to a function $W(z)$ of order m/n is a rational function

$$(A4.1) \qquad [m/n]_W(z) = \frac{U(z)}{V(z)} = \frac{\sum_{k=0}^m u_k z^k}{1+\sum_{k=1}^n v_k z^k},$$

where the coefficients u_k and v_k are chosen so that its Taylor series agrees with the Taylor expansion of $W(z)$ to the highest possible order. In other words

$$(A4.2) \qquad \left.\frac{\partial^k}{\partial z^k}[m/n]_W(z)\right|_{z=0} = \left.\frac{\partial^k}{\partial z^k}W(z)\right|_{z=0},$$

for $k = 0, \ldots, m+n+1$. It the $[m/n]_W$ Padé approximant exists then the u_k and v_k are uniquely determined.

For example, if $W(w) = e^{-z}$, solving

$$(A4.3) \qquad u_0 + u_1 z = \left(1 - z + \tfrac{1}{2}z^2\right)\left(1+v_1 z\right)$$

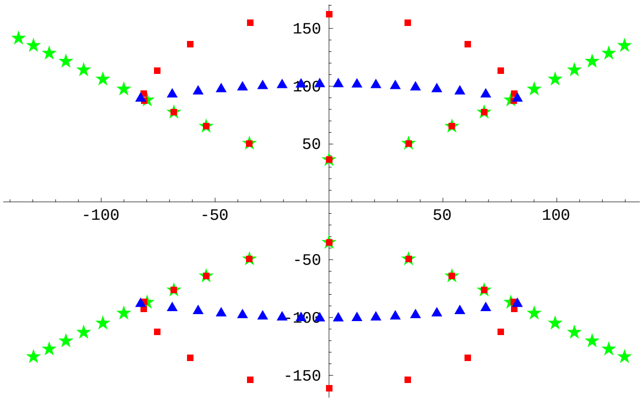

Figure I: Smallest exact zeros (★) and plot of zeros (■) and poles (▲) of approximation to $L(z) = 1 - qK(z)$ ($q = 0.1$) with [24/36] Padé approximant to $K(z)$.

by equating coefficients of powers of z, gives

$$[1/1]_W(z) = \frac{1 - \frac{z}{2}}{1 + \frac{z}{2}}. \tag{A4.4}$$

Such approximations usually give better approximation than the truncated Taylor series itself. Moreover, they often give a good approximation far outside the region in which the Taylor series converges. Effectively, a Padé approximant gives an analytic continuation of the power series beyond its radius of convergence by localizing singularities in the extended domain.

For any value of $m \leqslant n$, obtaining the roots of $V(z)$ as $\{p_k\}_{k=1}^n$ and those of $V(z) - U(z)$ as $\{r_k\}_{k=1}^n$ yields an approximation of order n/n in factorized form,

$$1 - e^{-z} \approx \prod_{k=1}^{n} \frac{z - r_k}{z - p_k}. \tag{A4.5}$$

A rational approximation of order $2n/2n$ to

$$L(z) = 1 - \exp\left\{\ln q + i\mu z - \sigma^2 z^2/2\right\}, \tag{A4.6}$$

with arbitrary $q \in \mathbb{C}$, $\mu, \sigma \in \mathbb{R}$, is then given with roots

$$\left\{\frac{i}{\sigma^2}\left(\mu \pm \sqrt{\mu^2 - 2\sigma^2\left(\ln q + r_k\right)}\right)\right\}_{k=1}^{n} \tag{A4.7}$$

and poles

$$\left\{\frac{i}{\sigma^2}\left(\mu \pm \sqrt{\mu^2 - 2\sigma^2\left(\ln q + p_k\right)}\right)\right\}_{k=1}^{n}. \tag{A4.8}$$

An example of the exact zeros given by (A3.3) closest to the real line plotted against the corresponding zero and poles of the approximation to $L(z) = 1 - qK(z)$ (using $q = 0.1$) with [24/36] Padé approximant to $K(z)$ is given in Figure I.

A5. **Approximation of Modified Wiener-Hopf Factors** $J_{\pm}(z)$**.** Suppose the Padé approximation procedure results in approximation $L^a_{\pm}(z)$ to $L_{\pm}(z)$ with a finite set of zeros $\Xi^{\mp}$ and poles $\Upsilon^{\mp}$ (in lower/upper half-plane $\Im z \lessgtr 0$), that is

$$L_{\pm}(z) \approx L^a_{\pm}(z) = \prod_{r_i \in \Xi^{\mp}} \prod_{p_j \in \Upsilon^{\mp}} \frac{z - r_i}{z - p_j}. \tag{A5.1}$$

Using these approximations, deformation of the integration contours in (2.57) upwards and those in (2.58) downwards results in the summation of the residues associated with the zeros in $\Xi^{\mp}$. This yields an implicit approximation $J^a_{\pm}(z)$ to $J_{\pm}(z)$ expressed as follows:

$$\frac{J^a_-(z)}{L^a_-(z)} - \frac{1}{2\pi i} \sum_{r_j \in \Xi^+} \frac{e^{i(u-d)r_j} J^a_+(r_j)}{(r_j - z)} R_-(r_j) = -\frac{1}{2\pi i} \sum_{r_j \in \Xi^+} \frac{e^{-idr_j}}{(r_j - z)} R_-(r_j), \tag{A5.2}$$

$$\frac{J^a_+(z)}{L^a_+(z)} - \frac{1}{2\pi i} \sum_{r_j \in \Xi^-} \frac{e^{-i(u-d)r_j} J^a_-(r_j)}{(r_j - z)} R_+(r_j) = -\frac{1}{2\pi i} \sum_{r_j \in \Xi^-} \frac{e^{-iur_j}}{(r_j - z)} R_+(r_j), \tag{A5.3}$$

where $R_{\pm}(r_j)$ is the residue of $1/L^a_{\pm}(z)$ at $z = r_j \in \Xi^{\mp}$. Considering (A5.2) at the points $z = r_k \in \Xi^+$ and (A5.3) at the points $z = r_k \in \Xi^-$, the equations may be solved by simple linear algebra to give $J^a_{\pm}(r_k)$ for $r_k \in \Xi^{\mp}$. An approximation to $J_{\pm}(z)$ at other points ($z \neq r_k \in \Xi^{\mp}$) can be obtained by interpolation, i.e. substituting $J^a_{\pm}(r_k)$ back into (A5.2) and (A5.3). That said, approximation of double-barrier option prices from (3.25) using this approximation $L^a_{\pm}(z)$ to $L_{\pm}(z)$ and the corresponding approximation $J^a_{\pm}(z)$ to $J_{\pm}(z)$ does not require this interpolation. Deformation of the integration contour in (3.25) (downwards for the part of the integrand involving $J^a_-(z)$ and upwards for that involving $J^a_+(z)$) results in a sum of terms involving $J^a_{\pm}(z)$ evaluated at the zeros of $L^a_{\pm}(z)$, that is

$$C_{DB} \approx \tilde{C}_{DB} + \frac{e^{-rT} \mathcal{Z}_N^{-1}}{2\pi} \left[\sum_{r_j \in \Xi^-} \hat{w}^{(\mathrm{ud})}(-r_j) \frac{e^{idr_j} J^a_-(r_j)}{L^a_-(r_j)} R_+(r_j) - \sum_{r_j \in \Xi^+} \hat{w}^{(\mathrm{ud})}(-r_j) \frac{e^{iur_j} J^a_+(r_j)}{L^a_+(r_j)} R_-(r_j) \right]. \tag{A5.4}$$

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